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BOUNDARIES OF RANDOM WALKS ON GRAPHS AND GROUPS WITH INFINITELY MANY ENDS

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Dedicated to the memory of Dr. Konrad Kovar

ABSTRACT

Consider an irreducible random walk $\{Z_n\}$ on a locally finite graph $\mathbf{G}$ with infinitely many ends, and assume that its transition probabilities are invariant under a closed group Γ of automorphisms of $\mathbf{G}$ which acts transitively on the vertex set. We study the limiting behaviour of $\{Z_n\}$ on the space $\mathbf{\Omega}$ of ends of $\mathbf{G}$. With the exception of a degenerate case, $\mathbf{\Omega}$ always constitutes a boundary of Γ in the sense of Furstenberg, and $\{Z_n\}$ converges a.s. to a random end. In this case, the Dirichlet problem for harmonic functions is solvable with respect to $\mathbf{\Omega}$. The degenerate case may arise when Γ is amenable; it then fixes a unique end, and it may happen that $\{Z_n\}$ converges to this end. If $\{Z_n\}$ is symmetric and has finite range, this may be excluded. A decomposition theorem for $\mathbf{\Omega}$, which may also be of some purely graph-theoretical interest, is derived and applied to show that $\mathbf{\Omega}$ can be identified with the Poisson boundary, if the random walk has finite range. Under this assumption, the ends with finite diameter constitute a dense subset in the minimal Martin boundary. These results are then applied to random walks on discrete groups with infinitely many ends.

§1. Introduction

Throughout this paper, we assume that $\mathbf{G} = (\mathbf{X}, \mathbf{E})$ is a vertex-transitive graph which is locally finite, infinite and connected. The edges in $\mathbf{E}$ are unoriented, there are no multiple edges. Furthermore, we assume that $\mathbf{G}$ has more than two ($\equiv$infinitely many) *ends*. Briefly spoken this means that by removing a finite piece, $\mathbf{G}$ can be split into three or more infinite components. Typical examples are homogeneous trees, but also Cayley graphs of groups

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which are free products with amalgamation or HNN-extensions. The *space* $\mathbf{\Omega}$ *of ends of* $\mathbf{G}$ arises as the boundary in a natural compactification of $\mathbf{G}$, see Freudenthal [Fr] and §2 below.

Recall that an automorphism of $\mathbf{G}$ is a bijection of $\mathbf{X}$ onto itself which preserves the adjacency relation of $\mathbf{G}$. With the topology of pointwise convergence, the group AUT($\mathbf{G}$) of all automorphisms of $\mathbf{G}$ is a locally compact, totally disconnected Hausdorff group. In the sequel, Γ will always be a closed subgroup of AUT($\mathbf{G}$) which is assumed to act *transitively* on the vertex set $\mathbf{X}$.

The principal object of our study will be a random walk on $\mathbf{G}$, that is, a homogeneous Markov chain Z_n, $n = 0, 1, 2, \dots$, with state space $\mathbf{X}$ and transition probabilities

$$p(x, y) = \Pr[Z_{n+1} = y \mid Z_n = x], \qquad x, y \in \mathbf{X},$$

which is adapted to the graph structure by the following hypotheses.

(i) *Group invariance*: $p(\gamma x, \gamma y) = p(x, y)$ for every $\gamma \in \Gamma$, $x, y \in \mathbf{X}$, and
(ii) *irreducibility*: if $x, y \in \mathbf{X}$ then $p^{(n)}(x, y) > 0$ for some $n \geqq 1$.

Here, $p^{(n)}(x, y) = \Pr[Z_n = y \mid Z_0 = x]$. The most natural example is the *simple random walk* on $\mathbf{G}$, where $p(x, y) = 1/D$ if x and y are neighbours (D denotes the common vertex degree), and $p(x, y) = 0$ otherwise. However, *a priori* we even allow that $\{y \mid p(x, y) > 0\}$ is infinite, although we shall restrict ourselves to finite range later on.

As $\mathbf{G}$ is vertex-transitive and has infinitely many ends, assumptions (i) and (ii) imply that $\{Z_n\}$ is *transient*, see Soardi and Woess [SW]. In other words, if $d(x, y)$ denotes the natural distance in $\mathbf{G}$ between vertices x and y, then

$$\Pr[d(Z_n, x) \to \infty \mid Z_0 = x] = 1 \tag{1.1}$$

for every $x \in X$. Hence, it is natural to ask whether Z_n converges with probability 1 to some random end of $\mathbf{G}$, and, in the affirmative case, to study the corresponding limiting probability ν on $\mathbf{\Omega}$. The questions are closely linked with the action of Γ on $\mathbf{\Omega}$.

We first describe in some detail the space of ends and the action of Γ (§2). If Γ fixes a finite subset of $\mathbf{\Omega}$ then this must be a singleton. This happens if and only if Γ is amenable (Theorem 2.3). We then show that Z_n, starting at some reference vertex o, can be described in terms of a random walk $S_n = X_1 \cdots X_n$, $n = 0, 1, 2, \dots$, on Γ which is governed by some Borel probability measure μ whose support generates Γ. Thus, we are led to the study of random walks on Γ of this type (§3). If Γ fixes no point of $\mathbf{\Omega}$, then with probability one, $S_n o$

converges to some end, and the limiting distribution is a continuous measure (Theorem 3.3 and Corollary 3.5). This is proved as the theorem of Cartwright and Soardi [CS], who have adapted results from two deep papers by Furstenberg [F1], [F2] to the situation of *trees*. The same is true when Γ fixes an end but μ admits a stationary measure ν on $\mathbf{\Omega}$ which is continuous (Corollary 3.6).

In §4, we derive our essential graph-theoretic result, which is used to obtain a more detailed description of the pair $(\mathbf{\Omega}, \nu)$. We apply a theorem of Dunwoody [Du] to obtain a decomposition of $\mathbf{\Omega}$ into one, two or three Borel sets, respectively, with useful properties (Theorem 4.1): $\mathbf{\Omega}^{(0)}$, the first part, consists of ends with bounded *finite diameter* only, and $\nu(\mathbf{\Omega} \setminus \mathbf{\Omega}^{(0)}) = 0$ (Corollary 4.2). Thus $S_n o$ converges with probability one to a random end with finite diameter whenever μ admits a continuous stationary measure on $\mathbf{\Omega}$ (the "nondegenerate" case).

We then return to the study of the random walk Z_n on $\mathbf{G}$ and the corresponding harmonic functions (§§5–9). In the nondegenerate case, the Dirichlet problem can be solved with respect to $\mathbf{\Omega}$ (§5, Theorem 5.2). If in addition to properties (i) and (ii), $\{Z_n\}$ has *finite range*, then it always converges a.s. in the end topology. In §6, we study the degenerate case when the limit is a deterministic end fixed by Γ. If the transition operator has norm < 1 on $l^2(\mathbf{X})$ (in particular, when the random walk is symmetric) then this can be excluded even for amenable Γ (Theorem 6.3).

Still assuming finite range in §§7 and 8, we use the results of §4 to prove that the pair $(\mathbf{\Omega}, \nu)$ coincides (up to mod-0-isomorphism) with the *Poisson boundary* of the random walk, so that every bounded harmonic function has a unique integral representation over $\mathbf{\Omega}$ (Theorem 7.1). The proof is based on the results of Picardello and Woess [PW1], [PW2] concerning *Martin boundaries* ($\equiv$ the cone of positive harmonic functions). In §8, the question is studied how far $\mathbf{\Omega}$ is from the Martin boundary of $\{Z_n\}$. It is proved that the set of ends with finite diameter constitutes a dense subset of the minimal Martin boundary in the corresponding topology (Theorem 8.2). However, ends with infinite diameter may still correspond to large portions of the Martin boundary.

Finally, in §9, we discuss how our results apply to random walks on discrete groups viewed in terms of their Cayley graphs. If Γ is discrete and has infinitely many ends, then Stallings' [St] structure theorem yields that the set $\mathbf{\Omega}^{(0)}$ exhibited in §4 can be represented as the set of *infinite words* with respect to some decomposition of Γ. Thus, every irreducible random walk converges almost surely to a random infinite word, and in case of finite range the set of infinite words can be identified with the Poisson boundary. This generalizes results of Derriennic [De] and Kaimanovich [Ka] known for free groups and free products, respectively.

On the whole, one may say that for a vertex-transitive graph with infinitely many ends, its "hyperbolic" features dominate in many respects, although in general such a graph is far from being hyperbolic in the sense of Gromov [Gr].

Inspiration for this paper came from Furstenberg's work [F1], [F2] and its application to trees by Cartwright and Soardi [CS] on the one hand, and on my joint work with M. A. Picardello concerning ends and Martin boundaries on the other. I am also indebted to to W. Imrich for pointing out the significance of Dunwoody's paper [Du] long before I have come to use it.

2. Ends and automorphisms

We briefly recall the construction of $\mathbf{\Omega}$. For every finite subset $U \subset \mathbf{X} \cup \mathbf{E}$, the family $\mathscr{C}_U$ of connected components in $\mathbf{G} \setminus U$ is finite by local finiteness of $\mathbf{G}$. With respect to set inclusion, the system of all $\mathscr{C}_U$, U finite, is directed. Its inverse limit is $\mathbf{\Omega}$. Thus, $\mathbf{X} \cup \mathbf{\Omega}$ becomes a compact Hausdorff space, $\mathbf{X}$ is discrete, open and dense and $\mathbf{\Omega}$ is compact. An end can be considered as an equivalence class of one-sided infinite paths without repeated vertices: two such paths are equivalent if for every finite $U \subset \mathbf{X} \cup \mathbf{E}$, all but finitely many of their vertices belong to the same component in $\mathscr{C}_U$. See Freudenthal [Fr] and Halin [H1], [H2] for fundamental facts concerning ends, and Woess [W2] for a more detailed graph-theoretical description close to the present setting.

If U is as above, then we implicitly include into every infinite set of $\mathscr{C}_U$ all its accumulation points in $\mathbf{\Omega}$. Thus, if $z \in \mathbf{X} \cup \mathbf{\Omega}$, $z \notin U$, then exactly one set in $\mathscr{C}_U$ contains z: we denote it by $C(U, z)$. The family of all $C(U, z)$, U finite not containing z, constitutes a neighbourhood basis at z consisting of open-closed sets. (Indeed, we may restrict ourselves to finite connected subgraphs U.) The *diameter* of an end ω is the minimal cardinal $k \leqq \aleph_0$ for which there exists a neighbourhood basis $\{C(U_i, \omega) \mid i \in I\}$ with finite $U_i \subset \mathbf{X}$, such that $\mathrm{diam}(U_i) \leqq k$ for all i. Here, the diameter refers to the discrete metric on $\mathbf{X}$ induced by $\mathbf{G}$. The set of ends with finite diameter will be denoted by $\mathbf{\Omega}_0$.

Every automorphism of $\mathbf{G}$ extends to $\mathbf{\Omega}$, and it is an easy exercise to show that the mapping

$$\mathrm{AUT}(\mathbf{G}) \times \mathbf{\Omega} \to \mathbf{\Omega}, \qquad (\gamma, \omega) \mapsto \gamma\omega \tag{2.1}$$

is jointly continuous.

The following lemma is due to Jung [Ju]; see also [SW] for a short proof. Recall that Γ acts vertex-transitively.

LEMMA 2.1. *If $U \subset \mathbf{X}$ is finite and such that $\mathscr{C}_U$ contains at least two infinite components, and if C is one of them, then there is an α in Γ such that $\alpha(U \cup C) \subset C$.*

In particular, α fixes exactly two ends ω_0, ω_1, their diameters are bounded by $\mathrm{diam}(U)$, $\omega_0 \in C$ and $\omega_1 \in \mathbf{\Omega} \setminus C$.

COROLLARY 2.2. *If $|\mathbf{\Omega}| \geqq 2$ then $\mathbf{\Omega}_0$ is dense in $\mathbf{\Omega}$.*

Indeed, we even have a dense subset of ends with uniformly bounded diameter. Similarly to the proof of Lemma 2.1, it is easy to show that by vertex-transitivity, $\mathbf{G}$ has either one, two or infinitely many ends [Fr], [H2]; here *we always assume that $\mathbf{\Omega}$ is infinite*. From [SW] (see also [H2], [W2]), we obtain the following result which shall be relevant in the sequel.

THEOREM 2.3. *Γ cannot fix a finite subset of $\mathbf{\Omega}$ other than a singleton. This happens if and only if Γ is amenable.*

PROOF. Suppose that $B \subset \mathbf{\Omega}$, $|B| = n$ $(0 < n < \infty)$, and $\gamma B = B$ for every $\gamma \in \Gamma$. Choose $\alpha \in \Gamma$ according to Lemma 2.1, and let ω_0, ω_1 be the two ends fixed by α. This is also the unique pair of ends fixed by $\gamma = \alpha^{n!}$. On the other hand, γ fixes B pointwise. Hence, we must have $B \subset \{\omega_0, \omega_1\}$. Now, Proposition 2 of [SW] (and its proof) show that Γ can fix only one out of $\{\omega_0, \omega_1\}$, and that this is equivalent with amenability of Γ by [W2]. ■

The following lemma parallels the one concerning trees in [CS] and is motivated by an argument concerning Fuchsian groups in [F1, Thm. 1.3].

LEMMA 2.4. *Let $\{\gamma_n\}$ be a sequence in Γ and $x \in \mathbf{X}$ such that* $\lim \gamma_n x = \omega_0$ *and* $\lim \gamma_n^{-1} x = \omega_1$ *with $\omega_0, \omega_1 \in \mathbf{\Omega}$. Then*

$$\lim_{n\to\infty} \gamma_n z = \omega_0 \qquad \text{for every } z \in \mathbf{X} \cup \mathbf{\Omega} \setminus \{\omega_1\},$$

and the convergence is uniform outside of every neighbourhood of ω_1.

PROOF. Let U, V be finite, connected subgraphs of $\mathbf{G}$. We show that there is an $N = N(U, V)$ such that $\gamma_n z \in C(U, \omega_0)$ for every $n \geqq N$ and every $z \in \mathbf{X} \cup \mathbf{\Omega}$, $z \notin C(V, \omega_1)$.

Let $r = \max\{d(x, v) \mid v \in V\}$. As $\gamma_n x \to \omega_0$, there is an index K such that $\gamma_n x \in C(U, \omega_0)$ and $d(\gamma_n x, U) > r$ for every $n \geqq K$. Hence, we also have

$\gamma_n V \subset C(U, \omega_0)$ for every $n \geqq K$. In the same way, there is an index L such that $\gamma_n^{-1} U \subset C(V, \omega_1)$ for every $n \geqq L$. Choose $N = \max\{K, L\}$. Let z be as above; we may assume $z \notin V$. If $n \geqq N$ then $U \subset \gamma_n C(V, \omega_1) = C(\gamma_n V, \gamma_n \omega_1)$, and the latter set does not intersect $\gamma_n C(V, z)$. Now, $\gamma_n V \cup \gamma_n C(V, z)$ is connected and does not intersect U: hence it is contained in one component in $\mathscr{C}_U$. By the choice of $N \geqq K$, this must be $C(U, \omega_0)$, and $\gamma_n z \in C(U, \omega_0)$ for every $n \geqq N$. ∎

We conclude this section with a few remarks concerning the topology of Γ. A neighbourhood basis of the identity is given by the family of pointwise stabilizers of finite sets in $\mathbf{X}$. These are open compact subgroups. We fix a reference vertex o and denote its stabilizer by Γ_o. If we choose and fix, for every $x \in \mathbf{X}$, an automorphism

$$\gamma_x \in \Gamma \qquad \text{with } \gamma_x o = x \qquad (\gamma_o = \iota, \ \text{the identity}), \tag{2.2}$$

then the $\gamma_y \Gamma_o \gamma_x^{-1}$, $x, y \in \mathbf{X}$, form a subbasis of the topology. Thus Γ is second countable.

3. Random walks on Γ

We can choose the left Haar measure $d\gamma$ on Γ such that $\int_{\Gamma_o} d\gamma = 1$, where o is our reference vertex. (For basic facts about integration on locally compact groups, see Hewitt and Ross [HR].) With the transition probabilities of Z_n, we associate a Borel measure μ on Γ, absolutely continuous with respect to $d\gamma$, by

$$\mu(d\gamma) = p(o, \gamma o) d\gamma. \tag{3.1}$$

This defines a probability measure, as

$$\int_\Gamma \mu(d\gamma) = \sum_{x \in \mathbf{X}} \int_{\gamma_x \Gamma_o} p(o, \gamma o) d\gamma = \sum_{x \in \mathbf{X}} p(o, x) \int_{\Gamma_o} d\gamma = 1.$$

Hence, we can define a probability space $(\Lambda, \mathscr{F}, \mathrm{Pr})$, where $\Lambda = \Gamma^{\mathbf{N}}$, $\mathscr{F}$ is the σ-Algebra generated by the cylinder sets $A_1 \times \cdots \times A_n \times \Gamma \times \Gamma \times \cdots$ ($n \geqq 1$, A_i Borel sets in Γ) and $\mathrm{Pr} = \mu^{\mathbf{N}}$. The projections $X_n : \Lambda \to \Gamma$ are independent and distributed according to μ. Thus, we can define the *right random walk*

$$S_n = \iota \cdot X_1 X_2 \cdots X_n, \qquad n \geqq 0 \tag{3.2}$$

on Γ; the distribution of S_n is the nth convolution power $\mu^{(n)}$ of μ. For further background, see Guivarc'h, Keane and Roynette [GKR].

LEMMA 3.1. *$\{S_n o\}$ is a model of the random walk $\{Z_n\}$ starting at o. In* *other words, $\{S_n o\}$ is a homogeneous Markov chain with transition probabilities $p(x, y)$, $x, y \in \mathbf{X}$.*

PROOF. First of all, if $\alpha \in \Gamma$ and $n \geqq 1$, then

$$\delta_\alpha * \mu^{(n)}(d\gamma) = p^{(n)}(\alpha o, \gamma o) d\gamma, \tag{3.3}$$

where δ_α is the point mass at α and $*$ denotes convolution. Indeed, if $n = 1$ then

$$\frac{d(\delta_\alpha * \mu)}{d\gamma}(\gamma) = \frac{d\mu}{d\gamma}(\alpha^{-1}\gamma) = p(o, \alpha^{-1}\gamma o) = p(\alpha o, \gamma o).$$

By induction,

$$\begin{aligned}\frac{d(\delta_\alpha * \mu^{(n+1)})}{d\gamma}(\gamma) &= \int_\Gamma \frac{d(\delta_\alpha * \mu^{(n)})}{d\gamma}(\beta) \frac{d\mu}{d\gamma}(\beta^{-1}\gamma) d\beta \\ &= \int_\Gamma p^{(n)}(\alpha o, \beta o) p(\beta o, \gamma o) d\beta \\ &= \sum_{x \in \mathbf{X}} \int_{\gamma_x \Gamma_o} p^{(n)}(\alpha o, x) p(x, \gamma o) d\beta = p^{(n+1)}(\alpha o, \gamma o).\end{aligned}$$

The statement of the lemma now follows from [GKR, p. 5, Remarque 6]. ∎

Furthermore, the following is easily deduced from the irreducibility hypothesis.

LEMMA 3.2. *The support of μ generates Γ as a semigroup.*

In view of Lemmas 3.1 and 3.2, we now consider a random walk $\{S_n\}$ on Γ, defined as in (3.2) by a probability measure μ which is not necessarily induced in the sense of (3.1) by a random walk on $\mathbf{G}$. We assume that *the support of μ generates Γ as a closed group*. If the support of μ generates Γ already as a closed semigroup, then we say that μ is *irreducible*.

Recall that convolution of μ with a measure ν on $\mathbf{\Omega}$ is defined by

$$\int_{\mathbf{\Omega}} f(\omega) \mu * \nu(d\omega) = \int_\Gamma \int_{\mathbf{\Omega}} f(\gamma\omega) \nu(d\omega) \mu(d\gamma), \qquad f \in C(\mathbf{\Omega});$$

in particular, we write $\gamma\nu$ for $\delta_\gamma * \nu$, $\gamma \in \Gamma$. If ν carries no positive point mass it is called *continuous*. The proof of the following result follows [CS].

THEOREM 3.3. *Suppose that there is a continuous probability* ν *on* $\boldsymbol{\Omega}$ *such that* $\mu * \nu = \nu$. *Then*

(a) ν *is unique with respect to these properties*,

(b) *for the random walk* S_n *governed by* μ,

$$\lim_{n\to\infty} S_n o \in \boldsymbol{\Omega}$$

exists almost surely in the end topology, and

(c) *if* $B \subset \boldsymbol{\Omega}$ *is a Borel set, then*

$$\nu(B) = \Pr\left[\lim_{n\to\infty} S_n o \in B\right].$$

PROOF. By [SW], Γ is nonamenable or nonunimodular. Hence, $\{S_n\}$ must be transient (see [GKR, Th. 47 and Th. 51]) and leaves every compact set with probability one. In particular, $d(S_n o, o) \to \infty$ almost surely. Furthermore, by [F2, Lemma 3.1 and Corollary], $S_n \nu$ converges a.s. to a probability on $\boldsymbol{\Omega}$. Let $\Lambda_1 \subset \Lambda$, $\Pr(\Lambda_1) = 1$, be the set where these two properties hold. Fix $\lambda \in \Lambda_1$, and let $m_\lambda = \lim S_n(\lambda)\nu$: for every $f \in C(\boldsymbol{\Omega})$,

$$\lim_{n\to\infty} \int_{\boldsymbol{\Omega}} f(S_n(\lambda)\omega)\nu(d\omega) = \int_{\boldsymbol{\Omega}} f(\omega) m_\lambda(d\omega). \tag{3.4}$$

Let $\{\gamma_k\}$ be a subsequence of $\{S_n(\lambda)\}$, such that $\{\gamma_k o\}$ and $\{\gamma_k^{-1} o\}$ converge in the end topology. As $d(S_n(\lambda)o, o) \to \infty$, the limits must be two ends ω_0 and ω_1, respectively. If $f \in C(\boldsymbol{\Omega})$, then Lemma 2.4 implies that $\lim f(\gamma_k \omega) = f(\omega_0)$ ν-almost surely, as $\nu(\omega_1) = 0$ by assumption. The dominated convergence theorem yields

$$\lim_{k\to\infty} \int_{\boldsymbol{\Omega}} f(\gamma_k \omega)\nu(d\omega) = f(\omega_0). \tag{3.5}$$

Comparing (3.4) and (3.5), we see that $m_\lambda = \delta_{\omega_0}$, a point mass. This must hold for every subsequence $\{\gamma_k\}$ of $\{S_n(\lambda)\}$ with $\{\gamma_k o\}$ and $\{\gamma_k^{-1} o\}$ convergent. By compactness, $S_n(\lambda) \to \omega_0 = \omega_0(\lambda)$. This proves (b).

As in [F2, p. 18] (in the proof of Prop. 3.3) one now obtains (c), which in turn implies (a). ■

By [F1, Lemma 1.2], there always is some probability ν on $\boldsymbol{\Omega}$ such that $\mu * \nu = \nu$; such a measure is called *stationary*. We want to know when there is a continuous one.

LEMMA 3.4. *Let* $B \subset \boldsymbol{\Omega}$ *be a closed set with the following properties*:

(a) *if* $\gamma \in \Gamma$ *and* $\gamma B \neq B$ *then* $\gamma B \cap B = \varnothing$,

(b) *there are infinitely many different* γB, $\gamma \in \Gamma$.

If ν *is a stationary probability measure for* μ *then* $\nu(B) = 0$.

PROOF. Let $a = \max\{\nu(\gamma^{-1}B) \mid \gamma \in \Gamma\}$. Suppose $a > 0$. Define

$$A = \bigcup \{\gamma^{-1}B \mid \nu(\gamma^{-1}B) = a\};$$

without loss of generality, we may assume $B \subset A$. By (a), A is a finite union of B-translates and hence closed. Furthermore,

$$b = \max\{\nu(\gamma^{-1}B) \mid \gamma^{-1}B \not\subset A\} < a.$$

Let $\alpha \in \operatorname{supp}(\mu)$, and suppose $\nu(\alpha^{-1}B) < a$. By (a), $\alpha^{-1}B \subset \mathbf{\Omega} \setminus A$, which is open. By (2.1) and (a), there is an open neighbourhood W of α in Γ such that $\beta^{-1}B \subset \mathbf{\Omega} \setminus A$ for every $\beta \in W$. Thus $\mu(W) > 0$, and

$$a = \nu(B) = \mu * \nu(B) = \int_W \nu(\beta^{-1}B)\mu(d\beta) + \int_{\Gamma \setminus W} \nu(\beta^{-1}B)\mu(d\beta)$$

$$\leqq b\mu(W) + a(1 - \mu(W)) < a,$$

a contradiction. Hence $\alpha^{-1}B \subset A$, and $\operatorname{supp}(\mu)$ fixes A.

Being the closed group generated by $\operatorname{supp}(\mu)$, also Γ fixes A. But this contradicts (b), and we must have $a = 0$. ∎

COROLLARY 3.5. *If* Γ *is nonamenable, then* μ *has a unique stationary probability* ν *on* $\mathbf{\Omega}$. *It is continuous, and* $\{S_n o\}$ *converges almost surely in the end topology with limiting distribution* ν.

PROOF. If in Lemma 3.4 we set $B = \{\omega\}$, $\omega \in \mathbf{\Omega}$, then (a) holds trivially, while (b) is a consequence of Theorem 2.3. As mentioned above, there is at least one stationary probability ν on $\mathbf{\Omega}$. By Lemma 3.4, ν must be continuous, and Theorem 3.3 applies. ∎

In the case when Γ is amenable, the situation is more complicated.

COROLLARY 3.6. *If* Γ *is amenable and fixes the end* ω_0, *then either*

(a) *the point mass at* ω_0 *is the unique stationary probability for* μ, *or*

(b) *there is a continuous stationary probability* ν, *and every stationary probability for* μ *on* $\mathbf{\Omega}$ *is a convex combination of* ν *and the point mass at* ω_0.

In the second case, $S_n o$ *converges in the end topology, with limiting distribution* ν.

PROOF. If δ_{ω_0} is not the only stationary probability for μ on $\mathbf{\Omega}$, then there must be another one, ν, such that $\nu(\omega_0) = 0$. By Theorem 2.3 and Lemma 3.4, ν must be continuous, and the statements follow from Theorem 3.3. ∎

In the sequel, the case of Corollary 3.6(a) will be called the *degenerate case*. We shall see below (§7) that both cases of Corollary 3.6 are possible for random walks with finite range.

We add another observation concerning the support of a stationary measure.

LEMMA 3.7. *If μ is irreducible and ν is a continuous stationary measure for μ on $\mathbf{\Omega}$, then the support of ν is the whole of $\mathbf{\Omega}$.*

PROOF. We consider ν as a measure on $\mathbf{G} \cup \mathbf{\Omega}$, not supported outside of $\mathbf{\Omega}$. We have to show that ν charges every infinite $C \in \mathscr{C}_U$ for every finite $U \subset \mathbf{X}$. We may assume in addition that U is large enough so that $\mathscr{C}_U$ has more than one infinite component. Let U and C be given.

By Lemma 2.1, there is $\alpha \in \Gamma$ with $\alpha(U \cup C) \subset C$. Let $\omega_0 \in C$, $\omega_1 \in \mathbf{\Omega} \setminus C$ be the two ends fixed by α. Then (see e.g. [H2], [W2]) $\alpha^n o \to \omega_0$ and $\alpha^{-n} o \to \omega_1$, and by Lemma 2.4 (or directly), $\{\alpha^{-n} C\}$ is an increasing sequence of sets with union $\mathbf{G} \cup \mathbf{\Omega} \setminus \{\omega_1\}$. As ν is continuous, $\nu(\alpha^{-n}C) \to 1$ and $\nu(\alpha^{-n}C) > 0$ for some $n > 0$. Without loss of generality, $\nu(\alpha^{-1}C) > 0$.

Consider $V = U \cup \{x\}$, with vertex x chosen arbitrarily in C. Let Γ_V be the subgroup of Γ which fixes V pointwise. Then $\Gamma_V \alpha$ is open-closed and $\gamma^{-1}C = \alpha^{-1}C$ for every $\gamma \in \Gamma_V \alpha$. By assumption, $\mu^{(k)}(\Gamma_V \alpha) > 0$ for some $k > 0$. Hence

$$\nu(C) = \int_\Gamma \nu(\gamma^{-1}C)\mu^{(k)}(d\gamma) \geqq \nu(\alpha^{-1}C)\mu^{(k)}(\Gamma_V \alpha) > 0. \qquad \blacksquare$$

We remark that $\mathbf{\Omega}$ can also be considered as a compactification of AUT(G): a sequence $\{\gamma_n\}$ in AUT(G) converges to $\omega \in \mathbf{\Omega}$ if $\gamma_n x \to \omega$ for some ($\equiv$all) $x \in \mathbf{X}$. Theorem 3.3 applies for any probability measure on AUT(G). If the closed group generated by the support of μ is noncompact and does not fix a finite set of ends ($\equiv$one or two; see [W2], [SW]), then μ has a unique stationary probability measure ν on $\mathbf{\Omega}$, ν is continuous and $(\mathbf{\Omega}, \nu)$ is a boundary of $(\mathrm{AUT}(\mathbf{G}), \mu)$ in the sense of [F2].

4. Cutting up $\mathbf{\Omega}$

In this section, we intend to continue our study of a random walk $\{S_n\}$ on Γ governed by a probability measure μ whose support generates Γ as a closed group. We are interested in further properties of the pair $(\mathbf{\Omega}, \nu)$ in the nondegenerate case, where ν is the limiting probability. For this purpose, we shall use the following decomposition therorem for the space of ends. It is based upon Dunwoody's [Du] approach to Stalling's structure theorem [St] and may be of some purely graph-theoretical interest by itself.

THEOREM 4.1. *The space of ends is a disjoint union* $\mathbf{\Omega} = \mathbf{\Omega}^{(0)}$, $\mathbf{\Omega} = \mathbf{\Omega}^{(0)} \cup \mathbf{\Omega}^{(1)}$ *or* $\mathbf{\Omega} = \mathbf{\Omega}^{(0)} \cup \mathbf{\Omega}^{(1)} \cup \mathbf{\Omega}^{(2)}$, *respectively, such that the parts have the following properties.*

(a) $\mathbf{\Omega}^{(0)} \subset \mathbf{\Omega}_0$, *there is a finite bound M such that* $\operatorname{diam}(\omega) \leqq M$ *for every* $\omega \in \mathbf{\Omega}^{(0)}$, *and* $\mathbf{\Omega}^{(0)}$ *is dense in* $\mathbf{\Omega}$.

(b) *If* $\mathbf{\Omega}^{(i)}$, $i \in \{1, 2\}$, *is nonvoid, then*

$$\mathbf{\Omega}^{(i)} = \bigcup \{\gamma B_i \mid \gamma \in \Gamma\},$$

where $B_i \subset \mathbf{\Omega}$ *is a closed set such that*

(b.1) *if* $\gamma \in \Gamma$ *and* $\gamma B_i \neq B_i$ *then* $\gamma B_i \cap B_i = \varnothing$, *and*

(b.2) $\{\gamma B_i \mid \gamma \in \Gamma\}$ *is countably infinite.*

PROOF. [Du] proves the existence of two infinite sets C_1, $C_2 = \mathbf{X} \setminus C_2$ of vertices of $\mathbf{G}$ with the following properties:

$$F = \{e \in \mathbf{E} \mid \text{one end of } e \text{ lies in } C_1 \text{ and one in } C_2\} \text{ is finite,} \tag{4.1}$$

and for every $\gamma \in \mathrm{AUT}(\mathbf{G})$, one of

$$C_1 \subset \gamma C_1, \quad C_1 \subset \gamma C_2, \quad C_2 \subset \gamma C_1, \quad C_2 \subset \gamma C_2 \tag{4.2}$$

holds. Implicitly, we identify each C_i with the subgraph of $\mathbf{G}$ which it induces, and we include into C_i all its accumulation points in $\mathbf{\Omega}$. Hence, $\mathbf{\Omega}$ is the disjoint union of $C_1 \cap \mathbf{\Omega}$ and $C_2 \cap \mathbf{\Omega}$. The C_i are not necessarily connected, but they are the respective unions of two parts of $\mathscr{C}_F$. We define

$$\mathscr{D} = \{\gamma C_i \mid \gamma \in \Gamma, i = 1, 2\}. \tag{4.3}$$

If $D = \gamma C_i \in \mathscr{D}$, then we write $D^* = \gamma C_{3-i}$ and $\overline{D} = \gamma(F \cup C_i)$. Note that $\mathscr{D}$ is countably infinite, as there is a two-to-one correspondence between $\mathscr{D}$ and the family $\{\gamma F \mid \gamma \in \Gamma\}$.

CLAIM 1. $\mathscr{D}$ *has the following properties.*

(I) *If* $D_1, D_2 \in \mathscr{D}$, *then one of* $D_1 \subset D_2$, $D_1^* \subset D_2^*$, $D_1 \subset D_2^*$ *or* $D_1^* \subset D_2$ *holds.*

(II) *If* $D_1, D_2 \in \mathscr{D}$, $D_1 \supset D_2$, *then*

$$[D_1, D_2] = |\{D \in \mathscr{D} \mid D_1 \supset D \supset D_2\}| - 1$$

is finite.

(III) *If* $D \in \mathscr{D}$ *then there are* $D_1, D_2 \in \mathscr{D}$ *such that* $D_1 \supset D \supset D_2$ *properly.*

PROOF OF CLAIM 1. (I) is a direct transcription of (4.2). (II) is proved in [Du, pp. 21–22]. (III) is indicated in [Du, p. 22]; we give some details:

We can find a finite connected subgraph U of $\mathbf{G}$ which contains F. Thus, $U \cup C_i$ is connected, $i = 1, 2$. For C_i, we can find $\gamma \in \Gamma$ such that $\gamma U \subset C_i$ and $\gamma U \cap U = \emptyset$. By connectedness, we must have $U \subset \gamma C_j$ for some $j \in \{1, 2\}$. Now, $\gamma(U \cup C_j^*)$ has empty intersection with $F \subset U$ and is contained in a component in $\mathscr{C}_F$. On the other hand, $\gamma U \subset C_i$. Hence $C_i \supset \gamma C_j^*$ properly. Thus, if $D \in \mathscr{D}$, then $D^* \supset D_1^*$ and $D \supset D_2$ properly for some $D_1, D_2 \in \mathscr{D}$, and (III) and Claim 1 are proved.

By (II) and (III), there are descending chains

$$(4.4) \qquad D_0 \supset D_1 \supset D_2 \supset \cdots, \qquad D_0 \in \{C_1, C_2\}, \qquad [D_{n-1}, D_n] = 1$$

of sets in $\mathscr{D}$.

CLAIM 2. *Each descending chain in* $\mathscr{D}$ *of type* (4.4) *defines a unique end* ω_0 *such that*

(IV) $\bigcap_n D_n = \{\omega_0\}$,

(V) $\operatorname{diam}(\omega_0) \leqq M = \operatorname{diam}(U)$, *where* U *is any finite connected subgraph of* $\mathbf{G}$ *containing* F, *and*

(VI) *different chains define different ends.*

PROOF OF CLAIM 2. Let $\{D_n\}$ be as in (4.4). It is clear that every $e \in F$ can be contained only in finitely many sets γF, $\gamma \in \Gamma$. Hence, there must be a subsequence $n_0 = 0, n_1, \ldots$, such that $D_{n_k} \supset \overline{D}_{n_{k+1}}$. If $x \in \mathbf{X} \setminus D_0$ and $y \in D_n$, $n \geqq n_k$, then every path in $\mathbf{G}$ which connects x and y must pass through a boundary edge of each of the D_l, $l \leqq n$, and there are more than n_k different ones. Hence $d(x, y) > n_k$ and

$$(4.5) \qquad d(x, D_n) \to \infty \qquad \text{as } n \to \infty.$$

Thus, $\bigcap D_n$ contains no vertex of $\mathbf{G}$. On the other hand, the D_n constitute a descending sequence of open-closed sets in the compact space $\mathbf{X} \cup \mathbf{\Omega}$ and must have nonvoid intersection. Therefore, there is some end $\omega_0 \in \bigcap D_n$.

Let $\omega \in \mathbf{\Omega}$, $\omega \neq \omega_0$. Then $C(V, \omega_0) \neq C(V, \omega)$ for some finite connected subgraph V of $\mathbf{G}$. Write $D_n = \gamma_n C_{i_n}$, where $\gamma_n \in \Gamma$ and $i_n \in \{1, 2\}$. By (4.5), $V \cap \gamma_n(U \cup C_{i_n}) = \emptyset$ for some n. As $\gamma_n(U \cup C_{i_n})$ is connected, it must lie in a component in $\mathscr{C}_V$. Furthermore, it contains ω_0, and we obtain $D_n \subset C(V, \omega_0)$. Thus $\omega \notin D_n$, and ω_0 is unique. This proves (IV).

If γ_n is as above, then by (4.5) every D_k contains all but finitely many $C(\gamma_n U, \omega_0)$, $n \in \mathbf{N}$. Thus, the latter constitute a neighbourhood basis at ω_0, and $\operatorname{diam}(\omega_0) \leqq \operatorname{diam}(U)$. This proves (V).

Finally, consider another descending chain $D_0' \supset D_1' \supset \cdots$ as in (4.4), with $\{\omega_0'\} = \bigcap D_n'$. Let n be the minimal index such that $D_n \neq D_n'$. If $n = 0$ then $\omega_0 \in D_0$ and $\omega_0' \in D_0^*$, so that $\omega_0 \neq \omega_0'$. Now suppose $n > 0$. Then $[D_{n-1}, D_n] = [D_{n-1}, D_n'] = 1$. In particular, we cannot have $D_n \subset D_n'$, $D_n' \subset D_n$ or $D_n^* \subset D_n'$. By (I), $D_n \cap D_n' = \emptyset$, and $\omega_0 \in D_n$ is different from $\omega_0' \in D_n'$, which proves (VI) and concludes the proof of Claim 2.

CONSTRUCTION OF $\mathbf{\Omega}^{(0)}$. We now define $\mathbf{\Omega}^{(0)}$ as the set of all ends given as in (IV) by a descending chain in $\mathscr{D}$ of type (4.4). To complete the proof of statement (a) of the theorem, what is left is to show that $\mathbf{\Omega}^{(0)}$ is dense. Let U be as above, and let V be any other finite connected subgraph of $\mathbf{G}$ containing F. Observe that varying V with respect to these properties, $\bigcup \mathscr{C}_V$ gives a basis for the topology of $\mathbf{\Omega}$. Choose an infinite $C \in \mathscr{C}_V$. We have to find some $\omega_0 \in \mathbf{\Omega}^{(0)}$ in C. By transitivity, there is $\gamma \in \Gamma$ with $\gamma U \subset C$. Recall that C_1 and C_2 are the respective unions of two parts of $\mathscr{C}_F$. As V is connected, $V \subset \gamma C_i$ for some $i \in \{1, 2\}$. But $\gamma(U \cup C_i^*)$ is also connected and hence contained in C. In other words, $D = \gamma C_i^* \subset C$. On the other hand, as $F \subset V$, we must have $C_1 \supset C$ or $C_2 \supset C$. Thus, by (II) and (III), D is part of an infinite chain of type (4.4) and contains some $\omega_0 \in \mathbf{\Omega}^{(0)}$. Therefore $\mathbf{\Omega}^{(0)}$ is dense in $\mathbf{\Omega}$.

Next, we want to describe $\mathbf{\Omega} \setminus \mathbf{\Omega}^{(0)}$. For $D \in \mathscr{D}$, we define

(4.6) $B(D) = \{\omega \in \mathbf{\Omega} \mid \omega \in D, \omega \notin D' \text{ for every } D' \in \mathscr{D} \text{ with } D \supset D' \text{ properly}\}.$

CLAIM 3. *Let $D_1, D_2 \in \mathscr{D}$ and $B(D_1) \neq \emptyset$. Then $B(D_1) = B(D_2)$ if and only if $D_1 = D_2$ or $[D_2, D_1^*] = 1$. In any other case, $B(D_1) \cap B(D_2) = \emptyset$.*

PROOF OF CLAIM 3. We subdivide into four cases according to (I), assuming that $D_1 \neq D_2$.

Case 1. $D_1 \subset D_2$. Then $B(D_1) \subset D_1$, while $B(D_2) \subset D_2 \setminus D_1$, so that $B(D_1) \cap B(D_2) = \emptyset$.

Case 2. $D_1^* \subset D_2^*$. Then $D_2 \subset D_1$ and $B(D_1) \cap B(D_2) = \emptyset$ as above.

Case 3. $D_1 \subset D_2^*$. Then $D_1 \cap D_2 = \emptyset$ and hence also $B(D_1) \cap B(D_2) = \emptyset$.

Case 4. $D_1^* \subset D_2$. This is the only case when we have to be more careful.

Case 4.1. $[D_2, D_1^*] = 1$. We observe that in this case

$$(4.7) \qquad \{D \in \mathscr{D} \mid [D_1, D] = 1\} \cup \{D_1^*\} = \{D \in \mathscr{D} \mid [D_2, D] = 1\} \cup \{D_2^*\}.$$

Indeed, if $[D_2, D] = 1$ and $D \neq D_1^*$ then we cannot have $D_2 \supset D \supset D_1^*$ or

$D_2 \supset D_1^* \supset D$. Thus, by (I), either $D_1 \subset D$ or $D \subset D_1$. The first inclusion is impossible, as otherwise $\mathbf{\Omega} \subset D_1 \cup D_1^* \subset D_2$. Hence, $D \subset D_1$ properly. Let $D \subset D' \subset D_1$ and $[D_1, D'] = 1$. It must be $D' \neq D_2^*$, as $D \subset D_2$. Now an argument symmetrical to the one above shows that $D' \subset D_2$ properly. As $[D_2, D] = 1$, we must have $D' = D$ and $[D_1, D] = 1$. Thus, in (4.7), the set on the right is contained in the one on the left. By symmetry, equality holds. Obviously

$$B(D_i) = \{\omega \in D_i \mid \omega \notin D \text{ whenever } [D_i, D] = 1\},$$

and (4.7) yields $B(D_1) = B(D_2)$.

Case 4.2. $[D_2, D_1^*] > 1$. Then there is some $D \in \mathscr{D}$ such that $[D, D_1^*] = 1$ and $D_2 \supset D$ properly. By the above, $B(D_1) = B(D)$, and $B(D) \cap B(D_2) = \varnothing$ according to Case 1. Claim 3 is proved.

CLAIM 4. $\mathbf{\Omega} \setminus \mathbf{\Omega}^{(0)} = \bigcup\{B(D) \mid D \in \mathscr{D}\}$.

PROOF OF CLAIM 4. If $\omega \in \mathbf{\Omega} \setminus \mathbf{\Omega}^{(0)}$, then either $\omega \in C_1$, or $\omega \in C_2$. By (II) and (III), we can construct a *maximal* finite descending chain

$$D_0, D_1, \ldots, D_n \in \mathscr{D}, \qquad D_0 \in \{C_1, C_2\}, \qquad [D_{k-1}, D_k] = 1,$$

such that $\omega \in D_k$ for every k. Thus $\omega \in B(D_n)$.

Conversely, let $D \in \mathscr{D}$. If $D \subset C_1$ or $D \subset C_2$, then no infinite chain of type (4.4) can contain elements of $B(D)$, and $B(D) \subset \mathbf{\Omega} \setminus \mathbf{\Omega}^{(0)}$. Suppose that neither $D \subset C_1$ nor $D \subset C_2 = C_1^*$. By (I), $D^* \subset C_i$ properly for some $i \in \{1, 2\}$. By (II) and (III), there is a $D' \in \mathscr{D}$ with $D' \subset C_i$ and $[D', D^*] = 1$. According to Claim 3, $B(D) = B(D')$, which is contained in $\mathbf{\Omega} \setminus \mathbf{\Omega}^{(0)}$ by the above. This proves Claim 4.

We now prove statement (b) of the theorem. We set $B_i = B(C_i)$, $i = 1, 2$, and define $\mathbf{\Omega}^{(i)}$ as proposed. Every set in $\mathscr{D}$ is open-closed, so that

$$C_i \setminus \bigcup\{D \in \mathscr{D} \mid C_i \supset D \text{ properly}\}$$

is closed; B_i is the intersection of this difference set with $\mathbf{\Omega}$ and thus closed. Note that $\gamma B_i = B(\gamma C_i)$ if $\gamma \in \Gamma$. If B_i is nonvoid, then (b.1) holds by Claim 3, and the family in (b.2) is infinite by (III) and countable together with $\mathscr{D}$. Finally, by Claim 3, $\mathbf{\Omega}^{(1)} \cap \mathbf{\Omega}^{(2)}$ is nonvoid if and only if $B_1 \neq \varnothing$ and $\alpha B_1 = \beta B_2$ for some $\alpha, \beta \in \Gamma$, and $\mathbf{\Omega}^{(1)} = \mathbf{\Omega}^{(2)}$ in this case. ∎

We remark that in the proof we could also have used, with similar effort, the tree constructed in [Du]. Its directed edge set is $\mathscr{D}$; it is countable but not necessarily locally finite. The ends of this tree are in one-to-one correspondence with $\mathbf{\Omega}^{(0)}$, and if $B(D_1)$, $B(D_2)$ are nonvoid then $B(D_1)=B(D_2)$ if and only if D_1 and D_2 have the same origin as edges of the tree. Furthermore, $B(D)$ is fixed by the stabilizer in Γ of the origin of edge D in the tree. We also remark that the decomposition in Theorem 4.1 is not unique; it depends on the choice of the sets C_i in (4.1) and (4.2).

Corollary 4.2. *If ν is a stationary probability measure for μ, then the probability spaces $(\mathbf{\Omega}, \nu)$ and $(\mathbf{\Omega}^{(0)}, \nu)$ are isomorphic modulo a null set. In particular, if ν is nondegenerate, then $\{S_n o\}$ converges with probability one to a random end in $\mathbf{\Omega}^{(0)}$.*

Proof. It follows from Lemma 3.4 and Theorem 4.1(b) that $\nu(\gamma B_i)=0$ for every $\gamma\in\Gamma$, and (b.2) yields $\nu(\mathbf{\Omega}^{(i)})=0$ for $i=1,2$. ■

The significance of Corollary 4.2 relies on the fact that the ends in $\mathbf{\Omega}^{(0)}$ have bounded diameter: as we shall see below (§7), they are easy to handle because of this property. Furthermore, if $\mathbf{G}$ is the Cayley graph of a discrete group, then they can be described as "infinite words" (§9).

5. Harmonic functions and the Dirichlet problem

In this and the remaining sections we return to the setting of §1. Thus, $\{Z_n\}$ is a random walk on $\mathbf{G}$ with properties (i) and (ii), and μ is associated with it as in (3.1). If f is a real-valued function on $\mathbf{X}$, then we define

$$\mathscr{P}f(x)=\sum_y p(x,y)f(y), \tag{5.1}$$

whenever this series converges. We shall be interested in the linear space of *harmonic functions*,

$$\mathscr{H}=\{h:\mathbf{X}\to\mathbf{R}\mid \mathscr{P}h=h\}, \tag{5.2}$$

in particular in the bounded and the positive ones. In this section, we shall assume that we are not in the degenerate case of Corollary 3.6(a), i.e.

(iii) μ admits a continuous stationary measure ν on $\mathbf{\Omega}$.

The following is more or less obvious.

Lemma 5.1. *Suppose that assumptions* (i)–(iii) *hold.*

(a) *With probability one, the random walk $\{Z_n\}$, starting at $x\in\mathbf{X}$, converges to a random end $Z_\infty=Z_\infty[x]\in\mathbf{\Omega}$.*

(b) *The corresponding limiting probabilities ν_x on $\mathbf{\Omega}$,*

$$\nu_x(B) = \Pr[Z_\infty \in B \mid Z_0 = x], \qquad B \text{ a Borel set in } \mathbf{X}, \qquad x \in \mathbf{X},$$

satisfy $\nu_{\gamma o} = \delta_\gamma * \nu$ *for* $\gamma \in \Gamma$. *They are continuous, supported by the whole of* $\mathbf{\Omega}$ *and mutually absolutely continuous.*

(c) $\nu_x = \Sigma_y\, p(x, y)\nu_y$ *for every* $x \in \mathbf{X}$.

PROOF. As in Lemma 3.1, let $Z_n = S_n o$ be the random walk starting at o. From assumption (iii) and Theorem 3.3 we know that Z_n converges a.s. to $Z_\infty = Z_\infty[o]$ with limiting distribution $\nu_o = \nu$ given by (iii). By Lemmas 3.2 and 3.7, $\text{supp}(\nu_o) = \mathbf{\Omega}$. Now, if $x \in \mathbf{X}$ and $\gamma \in \Gamma$ with $\gamma o = x$, then by (i), $\{\gamma S_n o\}$ is a model for the random walk starting at x. As γ acts continuously on $\mathbf{X} \cup \mathbf{\Omega}$, $\gamma S_n o$ tends to $\gamma Z_\infty = Z_\infty[x]$ a.s., and (a) holds.

Furthermore, $\nu_x = \delta_\gamma * \nu_o$ is the corresponding limiting distribution. Thus ν_x is continuous and has support $\mathbf{\Omega}$. For $x, y \in \mathbf{X}$, we define

$$F(x, y) = \Pr[Z_n = y \text{ for some } n \geqq 0 \mid Z_0 = x]. \tag{5.3}$$

By irreducibility, $F(x, y) > 0$ for every $x, y \in \mathbf{X}$. If $B \subset \mathbf{\Omega}$ is a Borel set, then $\nu_x(B) \geqq F(x, y)\nu_y(B)$ by the strong Markov property. Thus, ν_y is absolutely continuous with respect to ν_x, and (b) is proved.

Again, (c) follows from the strong Markov property, factoring through the first step of the walk. ∎

We now can formulate the solution of the Dirichlet problem for $\mathbf{\Omega}$.

THEOREM 5.2. *Suppose that assumptions* (i)–(iii) *hold. If* h^* *is a continuous function on* $\mathbf{\Omega}$, *then it has a unique continuous extension to* $\mathbf{X} \cup \mathbf{\Omega}$ *which is harmonic on* $\mathbf{X}$.

PROOF. Let $x \in \mathbf{X}$, *choose* $\gamma \in \Gamma$ with $\gamma o = x$ and define

$$h(x) = \int_{\mathbf{\Omega}} h^*(\omega)\nu_x(d\omega) = \int_{\mathbf{\Omega}} h^*(\gamma\omega)\nu(d\omega). \tag{5.4}$$

(The second identity follows from Lemma 5.1(b).) By Lemma 5.1(c), h is harmonic. To prove continuity of the extended function, we have to show that $h(x_n) \to h^*(\omega_0)$ if $x_n \to \omega_0$ in the end topology. Let $\gamma_n \in \Gamma$, $\gamma_n o = x_n$. Then $\gamma_n o \to \omega_0$. Let $\{\gamma_{n_k}\}$ be a subsequence such that $\{\gamma_{n_k}^{-1} o\}$ converges in $\mathbf{X} \cup \mathbf{\Omega}$. The limit must be an end. By Lemma 2.4 and continuity of ν, $\gamma_{n_k}\omega \to \omega_0$ for ν-almost every $\omega \in \mathbf{\Omega}$, and by continuity of h^* on $\mathbf{\Omega}$,

$$h(x_{n_k}) = \int_{\mathbf{\Omega}} h^*(\gamma_{n_k}\omega)\nu(d\omega) \to h^*(\omega_0).$$

This is true for every subsequence with $\{\gamma_{n_k}^{-1}o\}$ convergent. By compactness, $h(x_n)$ tends to $h^*(\omega_0)$.

Finally, uniqueness follows from the maximum principle: If h' is another solution, then $h - h'$ extends continuously with value zero on $\mathbf{\Omega}$. Thus, $\max(h - h')$ and $\min(h - h')$ are attained on $\mathbf{X}$. By harmonicity and irreducibility, $h - h'$ must be constant and hence $= 0$. ∎

Observe that in this surprisingly simple proof we do not base ourselves on an *a priori* identification of $\mathbf{\Omega}$ with the Poisson or Martin boundary: the latter was necessary, for example, in the approach of [De] concerning finite range random walks on free groups (a case covered by Theorem 5.2) and in Series [Se] and Ancona [An]. On the basis of [F1], [F2], it is possible to state Theorem 5.2 in a more general and abstract setting. An analogue of Lemma 2.4 is an essential ingredient.

6. Random walks with finite range

Here and in the following two sections, we consider a random walk $\{Z_n\}$ with properties (i) and (ii). In addition, we assume

(iv) *finite range*: $s = \max\{d(x, y) \mid p(x, y) > 0\}$ is finite.

If $U \subset \mathbf{X}$ is finite, then we denote

$$U^s = \{x \in \mathbf{X} \mid d(x, U) \leqq s/2\}. \tag{6.1}$$

The following is then quite clear.

LEMMA 6.1. *Suppose* (i), (ii) *and* (iv) *hold. Then with probability one, the random walk* $\{Z_n\}$, *starting at* $x \in \mathbf{X}$, *converges to a random end* $Z_\infty = Z_\infty[x] \in \mathbf{\Omega}$.

PROOF. The proof is necessary only in the degenerate case but becomes more elementary anyhow.

We may assume that $x = o$. Indeed, everything said so far does not depend on the choice of the reference vertex. Let $(\Lambda, \mathscr{F}, \Pr)$ be the probability space of §3, so that $Z_n(\lambda) = S_n(\lambda)o$ for $\lambda \in \Lambda$. By (iv) and transience of Z_n, we can find $\Lambda_1 \subset \Lambda$ with $\Pr(\Lambda_1) = 1$, such that on Λ_1 we have $d(Z_{k+1}(\lambda), Z_k(\lambda)) \leqq s$ for every $k \geqq 0$, and $d(Z_n(\lambda), o) \to \infty$.

Let $\lambda \in \Lambda_1$ and assume that $\{Z_n(\lambda)\}$ has two different accumulation points $\omega, \omega' \in \mathbf{\Omega}$. Then $C(U, \omega) \neq C(U, \omega')$ for some finite $U \subset X$. If $Z_k(\lambda) \in C(U, \omega)$ and $Z_m(\lambda) \in C(U, \omega')$ for some $m > k$, then for some j, $k \leqq j \leqq m$, we must have $Z_j(\lambda) \in U^s$ (compare this with [PW1, Lemma 4]). Thus, $\{Z_n(\lambda)\}$ visits the finite set U^s infinitely often, a contradiction. By compactness, $Z_n(\lambda)$ tends to an end. ∎

From the above and Lemma 5.1, it is clear that the corresponding limiting distributions ν_x, $x \in \mathbf{X}$, are either all continuous with support $\mathbf{\Omega}$, or that $\nu_x = \delta_{\omega_0}$, where ω_0 is independent of x, so that Z_n converges to ω_0. The latter is possible only when Γ is amenable (Theorem 2.3), and our next aim is to study this particular case.

We have to introduce further notation. If $V \subset \mathbf{X}$, then we denote by T_V the first hitting time of Z_n in V, and define

$$a^V(x, y) = \Pr[Z_{T_V} = y,\, T_V < \infty \mid Z_0 = x]. \tag{6.2}$$

Furthermore, if U, $V \subset \mathbf{X}$, then we define the matrix

$$A(U, V) = (a^V(x, y))_{x \in U,\, y \in V}, \tag{6.3}$$

in particular, $\mathbf{a}(x, V) = A(\{x\}, V)$, a row vector. Note that $a^V(x, y) \leqq F(x, y)$ and that $A(U, V)$ is substochastic; compare with [PW1]. In addition, if B is a Borel set in $\mathbf{X} \cup \mathbf{\Omega}$, then let $\nu(V, B)$ denote the column vector $(\nu_x(B \cap \mathbf{\Omega}))_{x \in V}$.

Now suppose that Γ fixes the end ω_0. By Lemma 2.1, we can find a pair (U, α), such that

$$\begin{gathered} U \subset X \text{ is finite, } |\mathscr{C}_U| \geqq 2, \text{ and } \alpha \in \Gamma \\ \text{with } \alpha(U \cup C) \subset C, \text{ where } C = C(U, \omega_0). \end{gathered} \tag{6.4}$$

Proposition 6.2. *The random walk* $\{Z_n\}$ *converges a.s. to* ω_0 *if and only if for some* ($\equiv$ *every*) *pair* (U, α) *which satisfies* (6.4), $A(U^s, \alpha U^s)$ *is stochastic.*

Proof. First, suppose that $Z_\infty = \omega_0$ a.s., and let U, C and α be as in (6.4). As $\alpha\omega_0 = \omega_0$, all the sets $\alpha^n C$, $n \geqq 0$, contain ω_0. Thus, $\nu_x(\alpha^n C) = 1$ for every $x \in \mathbf{X}$ and $n \geqq 0$. Let $x \in U^s$.

Case 1. If $x \in \alpha U^s$ then $\mathbf{a}(x, \alpha U^s) = (\delta_x(y))_{y \in \alpha U^s}$ has row sum one.

Case 2. If $x \notin \alpha U^s$ then $x \notin \alpha C$ by (6.4) and the definition of U^s. We have $\Pr[Z_n \in \alpha C$ for some $n \mid Z_0 = x] = 1$. Furthermore, as already used above, Z_n must pass through αU^s in order to reach αC from x. In other words,

$$\Pr[T_{\alpha U^s} < \infty \mid Z_0 = x] = 1,$$

and $\mathbf{a}(x, \alpha U^s)$ has row sum one.

Thus, all row sums of $A(U^s, \alpha U^s)$ are equal to one.

Conversely, let $A(U^s, \alpha U^s)$ be stochastic for U, C and α as in (6.4). Write

$$A = A(U^s, \alpha U^s) = (a^{\alpha U^s}(x, \alpha y))_{x, y \in U^s}.$$

Then $A(U^s, \alpha^n U^s) = A^n$ and

$$(6.5) \qquad \nu(U^s, \alpha^n C) = A^n \nu(\alpha^n U^s, \alpha^n C) = A^n \nu(U^s, C)$$

by group invariance and the strong Markov property (recall that Z_n must pass through $\alpha^k U^s$ to reach $\alpha^k C$ from outside).

The vector $\nu(U^s, C)$ has strictly positive entries: either Z_n converges to ω_0, and all entries equal one, or Lemma 5.1 applies. By the Perron–Frobenius theorem in one of its many variants (see e.g. Seneta [Se]), there is a stochastic matrix A^∞ such that $A^{nd} \to A^\infty$ as $n \to \infty$, where $d \in \mathbf{N}$. (Indeed, with some additional effort one may show that $d = 1$.) Hence,

$$A^{nd}\nu(U^s, C) \to A^\infty \nu(U^s, C) = \mathbf{c},$$

a strictly positive vector. Now, $\{\alpha^n C\}$ is a decreasing sequence of sets constituting a neighbourhood basis at ω_0. Thus, by (6.5), $\lim_n \nu(U^s, \alpha^n C) = \mathbf{c}$, and every ν_x, $x \in U^s$, carries a positive mass at ω_0. Hence, the limiting distribution cannot be continuous, and by Lemmas 5.1 and 6.1, Z_n must converge to ω_0 a.s. ■

In practice, the condition of Proposition 6.2 is not always easy to apply (see §7 for a positive example). However, we can derive an easy-to-verify condition for the limiting distribution to be continuous.

Consider the operator $\mathscr{P}$ defined in (5.1). In view of assumptions (i), (ii) and (iv), it acts as a bounded operator on $l^2(\mathbf{X})$, with norm $\| \mathscr{P} \|$.

THEOREM 6.3. *If* $\| \mathscr{P} \| < 1$ *under assumptions* (i), (ii) *and* (iv), *then the limiting probabilities* ν_x, $x \in X$, *of* Z_n *on* $\mathbf{\Omega}$ *are continuous. This holds, in particular, when* $\mathscr{P}$ *is symmetric, that is, when* $p(x, y) = p(y, x)$ *for all* $x, y \in \mathbf{X}$.

PROOF. If Γ is nonamenable, then every ν_x is continuous by Theorem 3.3 and Lemma 5.1, without assuming $\| \mathscr{P} \| < 1$.

Suppose that Γ is amenable and fixes $\omega_0 \in \mathbf{\Omega}$. We have to show that $\nu_x(\omega_0) = 0$ for some $x \in \mathbf{X}$. Consider the *Green kernel*

$$(6.6) \qquad G(x, y) = \sum_{n=0}^{\infty} p^{(n)}(x, y), \qquad x, y \in \mathbf{X}.$$

Our assumption yields that for $y \in \mathbf{X}$,

$$G(\cdot, y) = (\mathscr{I} - \mathscr{P})^{-1}\delta_y \in l^2(\mathbf{X}).$$

In particular, G vanishes at infinity: if $\varepsilon > 0$ then $G(x, y) < \varepsilon$ whenever $d(x, y) \geqq d_\varepsilon$ (and d_ε is independent of y by group invariance). Now clearly for any $V \subset \mathbf{X}$, $a^V(x, y) \leqq F(x, y) \leqq G(x, y)$, where $F(x, y)$ is as in (5.3).

Choose U, C and α as in (6.4). By the above, the entries of $A(U^s, \alpha^n U^s)$ tend to zero as $n \to \infty$, so that (6.5) yields

$$\nu(U^s, \{\omega_0\}) \leqq \nu(U^s, \alpha^n C) \to \mathbf{0},$$

the zero vector. Finally, it is proved in [SW, Thms. 2 and 3] that $\| \mathscr{P} \| < 1$ if $\mathscr{P}$ is symmetric. ∎

As observed in the proof, Theorem 6.3 is of relevance only when Γ is amenable. In this case, [SW, Cor. 3] provides an easy-to-calculate formula for $\| \mathscr{P} \|$. However, we shall see below that the limiting distributions may be continuous even when $\| \mathscr{P} \| \geqq 1$. Indeed, the proof of Theorem 6.3 requires only that the Green kernel vanishes at infinity.

7. The Poisson boundary

We have seen that the pair $(\mathbf{\Omega}, \nu)$, with $\nu = \nu_o$, is a model for the points attained "at infinity" by the random walk $\{Z_n\}$ starting at o and for the corresponding limiting distribution: this is true if Γ is nonamenable (Corollary 3.5) or if $\{Z_n\}$ has finite range (Lemma 6.1) in addition to group invariance and irreducibility. In this section, we assume that hypotheses (i), (ii) and (iv) are valid. Using the graph-theoretical result of §4, we intend to show that up to mod-0-isomorphism, $(\mathbf{\Omega}, \nu)$ is the largest possible model of this type, that is, the *Poisson boundary*. Of the many equivalent definitions (see e.g. Kaimanovich and Vershik [KV]), we choose the one which is closest to our geometrical approach.

With $F(x, y)$ and $G(x, y)$ defined as in (5.3) and (6.6), the *Martin kernel* is

$$(7.1) \qquad K(x, y) = F(x, y)/F(o, y) = G(x, y)/G(o, y), \qquad x, y \in \mathbf{X}.$$

The *Martin boundary* $\mathscr{M}$ is the set of points added to $\mathbf{X}$ in the unique minimal compactification $\hat{\mathbf{X}}$ of $\mathbf{X}$ with the following properties: every $K(x, \cdot)$ extends continuously to $\hat{\mathbf{X}}$, and the family of all $K(\cdot, z)$, $z \in \hat{\mathbf{X}}$, is in one-to-one correspondence with $\hat{\mathbf{X}}$. For more details, see Kemeny, Snell and Knapp [KSK] and, with notation closer to the present setting, [PW1], [PW2]. Now, there is an $\mathcal{M}$-valued random variable $\hat{Z}_\infty$, such that in the topology of $\hat{\mathbf{X}}$,

$$\lim_{n\to\infty} Z_n = \hat{Z}_\infty \quad \text{almost surely,} \qquad \text{if } Z_0 = o. \tag{7.2}$$

Let $\hat{\nu} = \hat{\nu}_o$ denote the corresponding limiting distribution on $\mathcal{M}$. The pair (supp($\hat{\nu}$), $\hat{\nu}$) is the Poisson boundary; every *bounded* harmonic function has a unique integral representation

$$h(x) = \int_{\mathcal{M}} \hat{h}(z)K(x,z)\hat{\nu}(dz), \qquad \hat{h} \in L^\infty(\mathcal{M}, \hat{\nu}). \tag{7.3}$$

THEOREM 7.1. *Under assumptions* (i), (ii) *and* (iv), *we have the following.*

(a) *Either the measure* $\nu = \nu_o$ *is continuous and supported on the whole of* $\mathbf{\Omega}$, *the Poisson boundary is up to a null set isomorphic with* $(\mathbf{\Omega}, \nu)$, *and every bounded harmonic function h has a unique integral representation such that for every* $\gamma \in \Gamma$,

$$h(\gamma o) = \int_{\mathbf{\Omega}} h^*(\gamma\omega)\nu(d\omega), \qquad h^* \in L^\infty(\mathbf{\Omega}, \nu),$$

(b) *or the Poisson boundary can be identified with* $\{\omega_0\}$, $\omega_0 \in \mathbf{\Omega}_0$, *and every bounded harmonic function is constant. In this case,* Γ *must be amenable and fix* ω_0.

PROOF. Our hypotheses imply that $\mathcal{P}$ and $\{Z_n\}$, respectively, are uniformly irreducible in the sense of [PW1], [PW2] with respect to the metric of $\mathbf{G}$. In [PW2], it is required that $p(x,y) > 0$ only if $[x,y] \in \mathbf{E}$. To satisfy this hypothesis, we may draw an edge between $x, y \in \mathbf{X}$ whenever $1 \leqq d(x,y) \leqq s$. The new graph obtained in this way has the same vertex set and end compactification as $\mathbf{G}$ [Fr], and its automorphism group contains AUT($\mathbf{G}$). Hence, we may apply the results of [PW2], and the identity mapping on $\mathbf{X}$ extends uniquely to a continuous surjection

$$\pi : \hat{\mathbf{X}} \to \mathbf{X} \cup \mathbf{\Omega}, \qquad \pi(\mathcal{M}) = \mathbf{\Omega}. \tag{7.4}$$

Furthermore, again by [PW2], π is one-to-one on $\mathbf{\Omega}_0$, i.e.,

$$|\pi^{-1}(\{\omega\})| = 1 \qquad \text{for every } \omega \in \mathbf{\Omega}_0. \tag{7.5}$$

(Indeed, Theorem and Corollary in [PW2] are stated in a slightly different way, but what is proved is (7.5).) It is now clear that $\pi(\hat{Z}_\infty) = Z_\infty$ a.s., where Z_∞ is the limit in $\mathbf{\Omega}$ of Z_n starting at o, and that $\nu = \nu_o$ is the image of $\hat{\nu}$, i.e., $\nu = \hat{\nu}\pi^{-1}$.

We now apply the results of §4: $\mathbf{\Omega}^{(0)}$ is a Borel set in $\mathbf{\Omega}$, contained in $\mathbf{\Omega}_0$, and

$\nu(\mathbf{\Omega}^{(0)}) = 1$. Thus, for $\mathcal{M}^{(0)} = \pi^{-1}(\mathbf{\Omega}^{(0)})$ we have $\hat{\nu}(\mathcal{M}^{(0)}) = 1$, and the respective complements are null sets. By (7.5), π is one-to-one on $\mathcal{M}^{(0)}$.

If $\nu = \delta_{\omega_0}$, where ω_0 is fixed by Γ, then $\omega_0 \in \mathbf{\Omega}^{(0)}$ by Lemma 2.1, and Lemmas 5.1 and 6.1 show that statement (b) of the theorem holds.

If ν is continuous, then by the above π is a mod-0-isomorphism between $(\mathcal{M}, \hat{\nu})$ and $(\mathbf{\Omega}, \nu)$. To complete the proof of (a), first observe that everything said so far holds for arbitrary $u \in \mathbf{X}$ in the place of the reference vertex o. If $B \subset \mathbf{\Omega}$ is a Borel set, then by [KSK, Prop. 10-21] (using the fact that $\nu_o(\mathbf{\Omega}^{(0)}) = 1$),

$$\nu_u(B) = \int_{\pi^{-1}B} K(u, z)\hat{\nu}(dz) = \int_{B \cap \mathbf{\Omega}^{(0)}} K(u, \pi^{-1}\omega)\nu_o(d\omega).$$

Hence, Lemma 5.1(b) yields that

$$\frac{d\nu_u}{d\nu_o}(\omega) = K(u, \pi^{-1}\omega) \qquad \nu_o\text{-almost surely.}$$

Now, if $h \in \mathcal{H}$ is bounded and represented by $\hat{h}$ as in (7.3), then we set $h^*(\omega) = \hat{h}(\pi^{-1}\omega)$ on $\mathbf{\Omega}^{(0)}$ and $h^*(\omega) = 0$ on $\mathbf{\Omega} \setminus \mathbf{\Omega}^{(0)}$. We obtain

$$h(\gamma o) = \int_{\mathcal{M}} \hat{h}(z)K(\gamma o, z)\hat{\nu}(dz) = \int_{\mathbf{\Omega}} h^*(\omega)\nu_{\gamma o}(d\omega)$$
$$= \int_{\mathbf{\Omega}} h^*(\gamma\omega)\nu_o(d\omega). \qquad \blacksquare$$

We now give a simple example, based on [SW, Ex. 1], to illustrate the situation when Γ is amenable.

EXAMPLE 7.2. $\mathbf{G} = \mathbf{T}$, the homogeneous tree of degree $r + 1$, $r \geqq 2$.

Let ω_0 be an end of $\mathbf{T}$, and let Γ be the subgroup of AUT($\mathbf{G}$) which stabilizes ω_0. It acts vertex-transitively and has been studied by various authors (e.g. Betori, Faraut and Pagliacci [BFP]). One could consider Γ as a kind of "hyperbolic-affine" group. As in [SW], consider the ordering of the vertex set $\mathbf{X}$ induced by ω_0: in particular, if $x \in \mathbf{X}$ and $n \geqq 0$, then $x - n$ is the unique vertex on the geodesic from x to ω_0 at distance n from x. The most general type of random walk on $\mathbf{T}$ with properties (i), (ii) which is *nearest neighbour* (i.e., $p(x, y) > 0$ if and only if $d(x, y) = 1$) is given by

$$(7.6) \qquad p(x, x-1) = a, \quad p(x-1, x) = \frac{1-a}{r} \qquad \text{for } x \in \mathbf{X},$$

where $0 < a < 1$. Note that $\gamma(x - n) = \gamma x - n$ for $\gamma \in \Gamma$, so that each of

$F(x, x-1)$ and $F(x-1, x)$ are independent of x. One can calculate these probabilities with the methods of Cartier [Ca] or Gerl and Woess [GW]:

$$F(x, x-1) = a + (1-a)F(x, x-1)^2 \quad \text{and}$$

$$F(x-1, x) = \frac{1-a}{r} + (r-1)\frac{1-a}{r}F(x, x-1)F(x-1, x) + aF(x-1, x)^2$$

From these equations,

$$(7.7) \quad \begin{aligned} &F(x, x-1) = \frac{a}{1-a} \quad \text{and} \quad F(x-1, x) = \frac{1}{r}, \qquad \text{if } a \leqq \frac{1}{2}; \\ &F(x, x-1) = 1 \quad \text{and} \quad F(x-1, x) = \frac{1-a}{ra}, \qquad \text{if } a \geqq \frac{1}{2}. \end{aligned}$$

Thus, from Proposition 6.2 (or directly) we deduce that $\nu_x = \delta_{\omega_0}$ if $a \geqq \frac{1}{2}$, and that ν_x is continuous otherwise.

In other words, if $a \geqq \frac{1}{2}$, then $Z_n \to \omega_0$ almost surely, the Poisson boundary is trivial and all bounded harmonic functions are constant. Observe however that the Martin boundary coincides with the whole of $\mathbf{\Omega}$ by [Ca], so that there are many positive harmonic functions.

On the other hand, if $a < \frac{1}{2}$, then the limiting distributions are continuous, the Dirichlet problem admits solution, and the Poisson boundary is all of $\mathbf{\Omega}$.

With the formulae of [SW], one calculates

$$(7.8) \qquad \| \mathscr{P} \| = a\sqrt{r} + \frac{1-a}{\sqrt{r}},$$

which is less than one if and only if $a < 1/(\sqrt{r}+1)$, while by (7.7) the Green kernel (being multiplicative along geodesics) vanishes at infinity if and only if $a < \frac{1}{2}$. ∎

If $\{Z_n\}$ is the simple random walk on any vertex-transitive graph with infinitely many ends, then by Theorems 6.3 and 7.1, its Poisson boundary can be identified with the whole of $\mathbf{\Omega}$, and the corresponding Dirichlet problem is solvable. The simple random walk is of course invariant under the whole automorphism group, which in Example 7.2 is much larger than Γ. However, in [SW, Ex. 2] a graph is exhibited where the whole automorphism group fixes one of the infinitely many ends, while Theorems 6.3 and 7.1 still apply to the simple random walk.

8. More about the Martin boundary

Once more, we assume that $\{Z_n\}$ has properties (i) group invariance, (ii) irreducibility and (iv) finite range. In §7, in particular in Theorem 7.1 and its proof, we have collected some information concerning the Martin boundary. Recall that $\mathcal{M}$ is in correspondence with *positive* harmonic functions in a similar way as the Poisson boundary is with the bounded ones: if $h \in \mathcal{H}^+$ then there is a Borel measure ν^h on $\mathcal{M}$ such that

$$h(x) = \int_{\mathcal{M}} K(x, z)\nu^h(dz). \tag{8.1}$$

A priori, ν^h is not unique. Recall that $h \in \mathcal{H}^+$ is called *minimal* if, whenever $h_1 \leqq h$ on $\mathbf{X}$ for $h_1 \in \mathcal{H}^+$, then h_1/h is constant. It is known that every minimal positive harmonic function with $h(o) = 1$ is of the form $K(\cdot, z)$ with $z \in \mathcal{M}$. The corresponding subset $\mathcal{M}^*$ of $\mathcal{M}$ is the *minimal Martin boundary*. The integral representation (8.1) becomes unique when restricted to measures supported on $\mathcal{M}^*$ only. (See [KSK].)

Consider the surjection π of $\mathcal{M}$ onto $\mathbf{\Omega}$ described in (7.4). By (7.5), we may identify $\mathbf{\Omega}_0$ with $\pi^{-1}\mathbf{\Omega}_0$ (with some care concerning the topology), and $\mathbf{\Omega}_0 \subset \mathcal{M}^*$ by [PW2]. Now $\mathcal{M}^*$ may be considerably larger that $\mathbf{\Omega}_0$, see e.g. Woess [W1] and the final remarks in §9. On the other hand, $\mathbf{\Omega}_0$ is dense in $\mathbf{\Omega}$, and the aim of this section is to show that $\mathbf{\Omega}_0$ is also dense in $\mathcal{M}^*$ with respect to the Martin topology: thus, we know "almost all" of $\mathcal{M}$.

In order to prove this, we first need an auxiliary result. If $U \subset \mathbf{X}$ and $\gamma \in \Gamma$, then we write, as in §6,

$$\mathbf{a}(x, \gamma U) = (a^{\gamma U}(x, \gamma y))_{y \in U}. \tag{8.2}$$

By $\mathbf{0}$ ($\mathbf{1}$), we shall denote the vectors with all entries equal to zero (one) in the dimension according to the context.

LEMMA 8.1. *Let $U \subset \mathbf{X}$ be finite containing o, and let $\{\gamma_n\}$ be a sequence in Γ with $d(\gamma_n o, o) \to \infty$.*

(a) *There exists a subsequence $\{n_k\}$ such that for every $x \in \mathbf{X}$,*

$$\lim_{k \to \infty} \mathbf{a}(x, \gamma_{n_k} U)/F(x, \gamma_{n_k} o) = \mathbf{b}(x),$$

where $\mathbf{b}(x) = (b(x, y))_{y \in U} \neq \mathbf{0}$ is uniformly bounded in x.

(b) *If $\gamma_n o \to z \in \mathcal{M}$ in the Martin topology, then*

$$h(x) = b(x, y)K(x, z)$$

is harmonic for every $y \in U$.

(c) *If* $z \in \mathcal{M}^*$, *then* $\mathbf{b}(x) = \mathbf{b}$ *does not depend on* x.

PROOF. As $\gamma_n o \in \gamma_n U$, we can write

$$F(x, \gamma_n o) = \sum_{y \in U} a^{\gamma_n U}(x, \gamma_n y) F(\gamma_n y, \gamma_n o).$$

By (i) and (ii),

$$\sum_{y \in U} F(y, o) \frac{a^{\gamma_n U}(x, \gamma_n y)}{F(x, \gamma_n o)} = 1. \tag{8.3}$$

By (ii), $M = \max\{1/F(y, o) \mid y \in U\} < \infty$, and hence the entries of $\mathbf{a}(x, \gamma_n U)/F(x, \gamma_n o)$ are bounded by M uniformly in x and n. As $\mathbf{X}$ is countable, the usual diagonal method yields the existence of a subsequence $\{n_k\}$ as stated in (a), and $\mathbf{b}(x) \neq \mathbf{0}$ by (8.3).

If $x \in \mathbf{X}$ then choose n_k large enough such that $x \notin \gamma_{n_k} U$. Then

$$\mathbf{a}(x, \gamma_{n_k} U) = \sum_u p(x, u) \mathbf{a}(u, \gamma_{n_k} U),$$

and

$$\frac{\mathbf{a}(x, \gamma_{n_k} U)}{F(x, \gamma_{n_k} o)} K(x, \gamma_{n_k} o) = \sum_u p(x, u) \frac{\mathbf{a}(u, \gamma_{n_k} U)}{F(u, \gamma_{n_k} o)} K(u, \gamma_{n_k} o).$$

The summation on the right is finite. Hence, if $\gamma_n o \to z \in \mathcal{M}$ in the Martin topology, then the kernels converge to $K(\cdot, z)$, and passing to the respective limits, we get

$$\mathbf{b}(x) K(x, z) = \sum_u p(x, u) \mathbf{b}(u) K(u, z).$$

This proves (b).

Finally, if M is as above and $y \in U$, then $b(x, y) K(x, z) \leqq M K(x, z)$ for every $x \in \mathbf{X}$: if $z \in \mathcal{M}^*$, then $b(x, y)$ must be constant in x, and (c) follows. ■

We can now prove the main result of this section.

THEOREM 8.2. *Suppose that* (i), (ii) *and* (iv) *hold. Let* $\mathbf{\Omega}_0^-$ *denote the closure of* $\mathbf{\Omega}_0$ *in* $\mathcal{M}$. *Then* $\mathbf{\Omega}_0 \subset \mathcal{M}^* \subset \mathbf{\Omega}_0^-$, *where* $\mathcal{M}^*$ *is the minimal Martin boundary*.

PROOF. We have already mentioned that $\mathbf{\Omega}_0 \subset \mathcal{M}^*$ by [PW2].

Let $z \in \mathcal{M}^*$. Then there must be a sequence $\{\gamma_n\}$ in Γ, such that $K(x, \gamma_n o) \to$

$K(x, z)$ for every $x \in \mathbf{X}$. By (7.4), $\gamma_n o \to \omega$ in the end topology for some $\omega \in \mathbf{\Omega}$. We can find a countable neighbourhood base $\{C(V_k, \omega) \mid k \in \mathbf{N}\}$ of ω, such that each V_k is a finite connected subgraph of $\mathbf{G}$ and

$$C(V_k, \omega) \supset V_{k+1} \cup C(V_{k+1}, \omega) \qquad \text{for every } k.$$

(In the terminology of [PW2], $\{V_k\}$ is *contracting* towards ω.)

Now choose a finite $U \subset \mathbf{X}$, such that $o \in U$ and $\mathscr{C}_U$ contains at least two infinite components. Set

$$n_k = \min\{n \mid \gamma_n U \subset C(V_k, \omega)\}.$$

By connectedness, $V_k \subset \gamma_{n_k} C_{i_k}$ for some $C_{i_k} \in \mathscr{C}_U$, and we must have $\gamma_{n_k} C_{j_k} \subset C(V_k, \omega)$ for some infinite $C_{j_k} \in \mathscr{C}_U$. As $\mathscr{C}_U$ is finite, C_{j_k} must be the same infinitely often: without loss of generality (otherwise we pass to a sub-subsequence), we assume that $C_{j_k} = C_0 \in \mathscr{C}_U$ for all k, and C_0 is infinite. Furthermore, we may apply Lemma 8.1 and assume that

$$\lim_{k \to \infty} \mathbf{a}(x, \gamma_{n_k} U)/F(x, \gamma_{n_k} o) = \mathbf{b}(x) \neq \mathbf{0},$$

and $\mathbf{b}(x)$ does not depend on x by minimality of z. By Corollary 2.2, we can find $\omega_0 \in \mathbf{\Omega}_0 \cap C_0$.

Claim. *$\gamma_{n_k} \omega_0 \to z$ in the Martin topology.*

Proof of the Claim. If $W \subset \mathbf{X}$ is finite, write $F(W, y) = (F(x, y))_{x \in W}$. As a main step in [PW1] (which adapts to our situation, see [PW2]), the following is proved: if $\omega_0 \in \mathbf{\Omega}_0$ and W is as above, then there is a nonnegative, nonzero vector $F(W, \omega_0) = (F(x, \omega_0))_{x \in W}$ such that

$$\lim_{k \to \infty} \frac{1}{\langle F(W, y_k), \mathbf{1} \rangle} F(W, y_k) = F(W, \omega_0) \tag{8.4}$$
$$\text{whenever } y_k \to \omega_0 \text{ in the end topology.}$$

(Here, $\langle \cdot, \cdot \rangle$ denotes inner product.) In particular, if $x \in \mathbf{X}$ and k is large enough, then $\{o, x\} \cap \gamma_{n_k} C_0 = \varnothing$ and the random walk must pass through $\gamma_{n_k} U^s$ on the way from o or x to $\gamma_{n_k} C_0$. Thus (8.4) and group invariance yield

$$\begin{aligned} F(\{o, x\}, \gamma_{n_k} \omega_0) &= c \cdot A(\{o, x\}, \gamma_{n_k} U^s) F(\gamma_{n_k} U^s, \gamma_{n_k} \omega_0) \\ &= c \cdot A(\{o, x\}, \gamma_{n_k} U^s) \mathbf{f}, \end{aligned}$$

where $c > 0$ is the appropriate norming constant and $\mathbf{f} = F(U^s, \omega_0)$. Now, (8.4) applied to $W = \{o, x\}$ and $\gamma_{n_k}\omega_0 \in \mathbf{\Omega}_0$ yields

$$K(x, \gamma_{n_k}\omega_0) = \frac{F(\{o, x\}, \gamma_{n_k}\omega_0)_x}{F(\{o, x\}, \gamma_{n_k}\omega_0)_o} = \frac{\langle c \cdot \mathbf{a}(x, \gamma_{n_k}U^s), \mathbf{f} \rangle}{\langle c \cdot \mathbf{a}(o, \gamma_{n_k}U^s), \mathbf{f} \rangle}$$

$$= K(x, \gamma_{n_k}o) \frac{\langle \mathbf{a}(x, \gamma_{n_k}U^s)/F(x, \gamma_{n_k}o), \mathbf{f} \rangle}{\langle \mathbf{a}(o, \gamma_{n_k}U^s)/F(o, \gamma_{n_k}o), \mathbf{f} \rangle}.$$

As $k \to \infty$, the limit of this last expression is

$$K(x, z) \frac{\langle \mathbf{b}(x), \mathbf{f} \rangle}{\langle \mathbf{b}(o), \mathbf{f} \rangle} = K(x, z),$$

as $\mathbf{b}(x)$ is independent of x. Hence, $K(x, \gamma_{n_k}\omega_0)$ tends to $K(x, z)$ for every $x \in \mathbf{X}$, or, in other words, $\gamma_{n_k}\omega_0 \to z$ in the Martin topology. This concludes the proof. ∎

Of course, one would like to know more about the part $\mathcal{M} \setminus \mathbf{\Omega}_0$ of the Martin boundary. However, the examples of [W1] (see also §9 below) show that for an end with infinite diameter, its preimage $\pi^{-1}\{\omega\}$ in $\mathcal{M}$ may vary its size considerably in dependence of transition operator and graph structure.

9. Random walks on discrete groups

In this section, we discuss how the preceding results apply to discrete groups. Let Γ be a finitely generated group, and consider its Cayley graph $\mathbf{G} = C(\Gamma, S)$ with respect to some finite symmetric sets S of generators. The vertex set is $\mathbf{X} = \Gamma$ (so that we use ordinary letters for its elements), and the edges are $[x, xa] \equiv [xa, x]$ $(x \in \Gamma, a \in S)$. The space of ends of Γ then is, by definition, the space of ends of $\mathbf{G}$, and it is well known [Fr] that neither space nor end topology depend on the particular choice of S. Also, finiteness of the diameter of an end is independent of S.

Thus, we are considering a group Γ with infinitely many ends. As a closed subgroup of AUT($\mathbf{G}$), Γ acts by left multiplication, and it is well known that Γ is nonamenable, see e.g. [St], also [SW]. We may identify our reference vertex o with the group identity. If $\{Z_n\}$ is a random walk on $\mathbf{G} \equiv \Gamma$ with properties (i) and (ii), then it coincides with $\{S_n\}$ as defined in (3.2), given that it starts at o. We have

$$p(x, y) = \mu(x^{-1}y), \tag{9.1}$$

where μ is as in (3.1), and supp(μ) generates Γ as a semigroup. By nonamenability, Γ fixes no end. We summarize the results.

THEOREM 9.1. *Let Γ be a finitely generated group with infinite space of ends $\mathbf{\Omega}$, and let $\{Z_n\}$ be an irreducible random walk on Γ, governed by the probability measure μ. Then the following statements hold.*

(a) *Starting at $x \in \Gamma$, Z_n converges with probability one to a random end $Z_\infty = Z_\infty[x]$.*

(b) *The corresponding limiting distributions ν_x, $x \in \Gamma$, are continuous, supported on the whole of $\mathbf{\Omega}$ and mutually absolutely continuous. If o is the group identity, then ν_o is the unique stationary probability measure for μ on $\mathbf{\Omega}$.*

(c) *Every continuous function on $\mathbf{\Omega}$ has a unique extension to $\Gamma \cup \mathbf{\Omega}$ which is continuous in the end topology and harmonic on Γ with respect to μ.*

(d) *If μ has finite support, then $(\mathbf{\Omega}, \nu_o)$ coincides (up to* mod-0-*isomorphism) with the Poisson boundary of the random walk starting at o, and $\mathbf{\Omega}_0$, the set of ends with finite diameter, is a dense subset of the minimal Martin boundary.*

The most familiar instance of a group where this applies is the *free group* $\mathbf{F}$. There, ends can be identified with infinite reduced words over the free generators and their inverses. In this case, statement (a) of Theorem 9.1 is attributed, without proof, to G. A. Margulis in [KV]; a proof can be found in [CS]. Observe that $\mathbf{\Omega} = \mathbf{\Omega}_0$ for $\mathbf{F}$. If μ is finitely supported, then Martin and Poisson boundary coincide with $\mathbf{\Omega}$ (Dynkin and Malyutov [DM], Derriennic [De]). The same is true for groups with a free subgroup of finite index [PW1].

"Infinite words" are also present in any finitely generated group with infinitely many ends; they correspond to $\mathbf{\Omega}^{(0)}$ in Theorem 4.1 and thus depend on the decomposition of $\mathbf{\Omega}$. Indeed, by [St], Γ has a decomposition as an amalgamated free product or as an HNN-extension over a finite subgroup, see Lyndon and Schupp [LS, §IV.2] for details.

(A) Let Γ be the amalgamated free product of Γ_1 and Γ_2 over the common finite subgroup H. Then we may choose sets of representatives Y_i of Γ_i/H with $o \in Y_i$, $i = 1, 2$. Each $x \in \Gamma$ can then be written uniquely as

$$x = y_1 y_2 \cdots y_m h, \qquad \text{where } h \in H, \quad m \geqq 0,$$

(9.2)

$$y_j \in Y_{i_j} \setminus \{o\}, \qquad i_j \in \{1, 2\} \quad \text{and} \quad i_{j+1} \neq i_j.$$

An *infinite word* is then a sequence

(9.3) $$\omega_0 = y_1 y_2 y_3 \cdots,$$

where the y_i have the same properties as in (9.2). Here, ω_0 represents an end of Γ in the sense that $y_1 \cdots y_n \to \omega_0$ as $n \to \infty$. It is not hard to see that in Theorem 4.1 we may take for B_i the space of ends of Γ_i (the latter considered as a subset of Γ), $i = 1, 2$, and for $\mathbf{\Omega}^{(0)}$ the set of infinite words (9.3).

(B) Let $\Gamma = \langle \Gamma_0, t \mid ht = t\phi(h)\ \forall h \in H \rangle$ be an HNN-extension of Γ_0 over H, where H is a finite subgroup of the group Γ_0 and ϕ is an isomorphism from H onto $\phi(H) \leqq \Gamma_0$. Then we can choose sets of representatives Y_1 and Y_2, both containing o, of Γ_0/H and $\Gamma_0/\phi(H)$, respectively. Each $x \in \Gamma$ can be written uniquely as

(9.4) $x = y_0 t^{\varepsilon_0} y_1 t^{\varepsilon_1} \cdots y_{m-1} t^{\varepsilon_{m-1}} y_m$, where $m \geqq 0$, $y_m \in \Gamma_0$,
for $j = 0, \ldots, m-1$ either $\varepsilon_j = 1$, $y_j \in Y_1$ or $\varepsilon_j = -1$, $y_j \in Y_2$,
and there is no successive triplet of the form $t^{\varepsilon} o t^{-\varepsilon}$, $\varepsilon \in \{1, -1\}$.

An *infinite word* is then a sequence

(9.5) $$\omega_0 = y_0 t^{\varepsilon_0} y_1 t^{\varepsilon_1} y_2 t^{\varepsilon_2} \cdots$$

with the same properties as in (9.4). It represents an end of Γ in the sense that $y_0 t^{\varepsilon_0} \cdots y_n t^{\varepsilon_n} \to \omega_0$ as $n \to \infty$. In Theorem 4.1, we may choose the space of ends of Γ_0 in Γ as the set B_1 ($B_2 = \varnothing$), and the set of infinite words (9.5) as $\mathbf{\Omega}^{(0)}$.

COROLLARY 9.1. *Let $\{Z_n\}$ be an irreducible random walk on the finitely generated group Γ, governed by a probability measure μ. If Γ has infinitely many ends, i.e., Γ is an amalgamated free product or an HNN-extension over a finite subgroup, then Z_n converges almost surely to a random infinite word. If μ has finite support, then the Poisson boundary can be identified with the set of infinite words together with the corresponding limiting distribution.*

A particular case is that of a *free product* of two (or more) groups, without amalgamation (i.e., $H = \{o\}$). In his note [Ka], Kaimanovich has a more general result concerning the Poisson boundary on free products: instead of finite support, it is enough that μ has finite first moment with respect to the length m of x in (9.2). According to [Ka], the Poisson boundary then still is the set of infinite words, and it is stated that this holds not only when μ is irreducible, but whenever μ is not supported on one of the free factors. (Evidently, also the conjugates of the free factors have to be excluded.) [Ka] applies methods from ergodic theory, completely different from those used here.

In [W1], the Martin boundary is studied in detail when Γ is a free product of two groups and the support of μ contains only elements with $m = 1$ in representation (9.2); μ is assumed to be irreducible. A typical example is the simple random walk on the free product $\Gamma_1 * \Gamma_2$, where $\Gamma_1 = \Gamma_2 = \mathbf{Z}^d$, the d-dimensional grid. The Martin boundary consists of the set of infinite words plus parts which are "attached at infinity" to each of the Γ_i-cosets in Γ, $i = 1, 2$. If $d \geqq 2$, then each of these parts is homeomorphic with the unit sphere in the corresponding dimension, while it corresponds to one end of the Cayley graph.

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